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Empirical Quantum Mechanics

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Machida and Namiki developed a many-Hilbert-spaces formalism for dealing with the interaction between a quantum object and a measuring apparatus. Their mathematically rugged formalism was polished first by Araki from an operator-algebraic standpoint and then by Ozawa for Boolean quantum mechanics, which approaches a quantum system with a compatible family of continuous superselection rules from a notable and perspicacious viewpoint. On the other hand, Foulis and Randall set up a formal theory for the empirical foundation of all sciences, at the hub of which lies the notion of a manual of operations. They deem an operation as the set of possible outcomes and put down a manual of operations at a family of partially overlapping operations. Their notion of a manual of operations was incorporated into a category-theoretic standpoint into that of a manual of Boolean locales by Nishimura, who looked upon an operation as the complete Boolean algebra of observable events. Considering a family of Hilbert spaces not over a single Boolean locale but over a manual of Boolean locales as a whole, Ozawa's Boolean quantum mechanics is elevated into empirical quantum mechanics, which is, roughly speaking, the study of quantum systems with incompatible families of continuous superselection rules. To this end, we are obliged to develop empirical Hilbert space theory. In particular, empirical versions of the square root lemma for bounded positive operators, the spectral theorem for (possibly unbounded) self-adjoint operators, and Stone's theorem for one-parameter unitary groups are established.

0. INTRODUCTION

Some researchers on the foundations of quantum mechanics have discussed many-Hilbert-spaces formalisms, which are to be distinguished strictly from Everett's (1957) eccentric many-universes interpretation of quantum mechanics. Such attempts are gathering momentum, since macroscopic quantum phenomena are now commonplace and modern technology is realizing what were once reckoned as mere Gedanken experiments. Among others, Machida and Namiki (1980) installed such a formalism so as to reconcile

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the microscopic and macroscopic viewpoints of the physical world, in particular, to settle decisively such famous paradoxes concerning the so-called reduction of wave packets as that of Einstein, Podolski, and Rosen (1935). Their mathematically somewhat rugged formalism was refined from the standpoint of operator algebras by Araki (1980), who considered macroscopic observables to be represented by the center of the algebra of microscopic observables. Finally, Ozawa (1986), remarking that the projection lattice of the center of a von Neumann algebra is a complete Boolean algebra and so the techniques designated Boolean-valued analysis after Takeuti (1978) are available, polished their ideas into *Boolean quantum mechanics*, from which we can recover Machida and Namiki's original picture through the process of averaging.

Foulis and Randall (1972; Randall and Foulis, 1973) have provided a formal theory for the foundation of all empirical sciences, in which the notion of a *manual of operations* plays a pivotal role. They regard an operation as the set of possible outcomes, which enjoys classical logic and classical statistics, and consider a manual of operations to be a family of partially overlapping operations. In Nishimura (1993b), by regarding an operation not as the set of possible outcomes, but as the complete Boolean algebra of observable events, we got the notion of a *manual of Boolean locales*, over which *empirical set theory* subsists.

Machida and Namiki's (1980) many-Hilbert-spaces formalism deals only with a family of Hilbert spaces over a single operation in the terminology of Foulis and Randall (1972; Randall and Foulis, 1973) or over a single Boolean locale in our terminology (Nishimura, 1993b). If we want to set up empirical foundations of quantum mechanics, we are obliged to treat a kind of many-Hilbert-spaces formalism over a manual of operations or rather over a manual of Boolean locales, so that Ozawa's (1986) Boolean quantum mechanics should be incorporated into *empirical quantum mechanics*, which is the main purpose of this paper.

The organization of the paper is as follows. After presenting preliminaries on such fundamental notions as a manual of Boolean locales in Section 1, we discuss empirical Hilbert spaces along the lines of Nishimura (1995b, 1996a, b, n.d.-a) in Section 2. Sections 3 and 4 are devoted to bounded and unbounded operators on empirical Hilbert spaces respectively. In particular, an empirical version of Stone's theorem is established. The final section deals with empirical quantum mechanics.

We close this introduction with a few notational and terminological comments. We denote by $\mathbf{N}$, $\mathbf{R}$, $\overline{\mathbf{R}}$, and $\mathbf{C}$ the sets of natural numbers, real numbers, extended real numbers (i.e., real numbers and $\pm\infty$), and complex numbers, respectively. A Hilbert space always means a complex Hilbert space. An isometric linear mapping of a Hilbert space into another is called a *Hilbert map*. A closed linear subspace of a Hilbert space H is called a *Hilbert* *subspace* of H. Given a category **A**, the classes of objects and morphisms of **A** are denoted respectively by Ob **A** and Mor **A**. If Mor **A** is a set, the category **A** is called *small.*

1. PRELIMINARIES

1.1. Universes

To dodge the famous paradoxes of set theory or to paper them over, the usage of a *universe* is a common practice in category theory. Roughly speaking, a universe is a well-behaved set closed under any standard operation of set theory. For the exact definition of a universe, see e.g., MacLane (1971, Chapter I, §6), Schubert (1972, §3.2), or Borceux (1994, Vol. 1, §1.1). The existence of a universe is disputable from the standpoint of axiomatic set theory, but we assume in this paper that there exists a universe V_0. The class of all sets is denoted by V. Sets of V_0 are called *small*$_0$. The adjective "small$_0$" is applied to structures whose underlying sets are small$_0$. By way of example, a category **C** is called small$_0$ if the class Mor **C** of morphisms of **C** is a small$_0$ set. We denote by **BEns** and **BEns**$_0$ the category of sets and functions and its full subcategory of small$_0$ sets, respectively.

1.2. Manuals of Boolean Locales

The category of complete Boolean algebras and complete Boolean homomorphisms is denoted by **Bool**. The dual category of **Bool** is denoted by **BLoc**. Its objects are called *Boolean locales*. If we regard a Boolean locale X as an object of **Bool**, it is denoted by $\mathscr{P}(\mathrm{X})$, though X and $\mathscr{P}(\mathrm{X})$ denote the same entity. The opposite of a morphism f: X $\to$ Y of **BLoc**, which is a complete Boolean homomorphism from $\mathscr{P}(\mathrm{Y})$ to $\mathscr{P}(\mathrm{X})$, is usually denoted by $\mathscr{P}(\mathrm{f})$. A morphism f of **BLoc** is called an *embedding* if $\mathscr{P}(\mathrm{f})$ is surjective. Two embeddings f: Y $\to$ X and g: Z $\to$ X with the same codomain X are said to be *equivalent* if there exists an isomorphism h: Y $\to$ Z in **BLoc** with $\mathrm{f} = \mathrm{g} \circ \mathrm{h}$. Given a Boolean locale X and $x \in \mathscr{P}(\mathrm{X})$, the morphism i_x: $\mathrm{X}|x \to \mathrm{X}$ is an embedding, where $\mathscr{P}(\mathrm{X}|x) = \mathscr{P}(\mathrm{X})|x = \{y \in \mathscr{P}(\mathrm{X}) | y \leq x\}$ and $\mathscr{P}(\mathrm{i}_x)(y) = x \wedge y$ for each $y \in \mathscr{P}(\mathrm{X})$. Any embedding into X is equivalent to i_x for a unique $x \in \mathscr{P}(\mathrm{X})$. A Boolean locale X is said to be trivial if $\mathscr{P}(\mathrm{X})$ is a trivial Boolean algebra, i.e., if $\mathscr{P}(\mathrm{X})$ consists of a single element. Since the category **Bool** is complete, the category **BLoc** is cocomplete.

Let $\mathfrak{M}$ be a small subcategory of the category **BLoc**. A diagram of **BLoc** is said to be *in* $\mathfrak{M}$ if all the objects and morphisms occurring in the diagram lie in $\mathfrak{M}$. Boolean locales X and Y in $\mathfrak{M}$ are said to be $\mathfrak{M}$*-orthogonal*, in notation $\mathrm{X} \perp_{\mathfrak{M}} \mathrm{Y}$, if there exists a coproduct diagram $\mathrm{X} \xrightarrow{\mathrm{f}} \mathrm{Z} \xleftarrow{\mathrm{g}} \mathrm{Y}$ of **BLoc** lying in $\mathfrak{M}$. A Boolean locale X in $\mathfrak{M}$ is said to be $\mathfrak{M}$*-maximal* if for any Boolean locale Y in $\mathfrak{M}$, $X \perp_{\mathfrak{M}} Y$ implies that Y is trivial. Boolean locales X and Y are said to be *$\mathfrak{M}$-equivalent*, in notation $X \simeq_{\mathfrak{M}} Y$, provided that for any Boolean locale Z in $\mathfrak{M}$, $X \perp_{\mathfrak{M}} Z$ iff $Y \perp_{\mathfrak{M}} Z$. Obviously $\mathfrak{M}$-equivalence is an equivalence relation among the Boolean locales in $\mathfrak{M}$. We denote by $[X]_{\mathfrak{M}}$ the equivalence class of X with respect to $\mathfrak{M}$-equivalence. A coproduct diagram $\{X_\lambda \xrightarrow{f_\lambda} X\}_{\lambda \in \Lambda}$ of **BLoc** lying in $\mathfrak{M}$ is called an *$\mathfrak{M}$-coproduct diagram* if for any diagram of the form $\{X_\lambda \xrightarrow{f'_\lambda} X'\}_{\lambda \in \Lambda}$ lying in $\mathfrak{M}$, the unique morphism $g: X \to X'$ of **BLoc** with $g \circ f_\lambda = f'_\lambda$ for all $\lambda \in \Lambda$ belongs to $\mathfrak{M}$, in which X is called an *$\mathfrak{M}$-coproduct* of X_λ's and is denoted by $\Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} X_\lambda$. If Λ is a finite set, say, $\Lambda = \{1, 2\}$, then such a notation as $X_1 \oplus_{\mathfrak{M}} X_2$ is preferred. If Λ is empty, $X = \Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} X_\lambda$ is no other than a trivial Boolean locale which is an initial object in $\mathfrak{M}$. In this case X is called an *$\mathfrak{M}$-trivial* Boolean locale. An embedding $f: X \to Y$ in $\mathfrak{M}$ is called an *$\mathfrak{M}$-embedding* if there exists an embedding $g: Z \to Y$ in $\mathfrak{M}$ such that the diagram $X \xrightarrow{f} Y \xleftarrow{g} Z$ is an $\mathfrak{M}$-coproduct diagram. In this case X is called an *$\mathfrak{M}$-sublocale* of Y. Given an $\mathfrak{M}$-sublocale Y of a Boolean locale X in $\mathfrak{M}$, the $\mathfrak{M}$-embedding of Y into X is equivalent in **BLoc** to the canonical embedding $i_x: X|x \to X$ for a unique $x \in \mathscr{P}(X)$, in which Y is denoted by X_x.

A *manual of Boolean locales* is a small subcategory $\mathfrak{M}$ of the category **BLoc** satisfying the following conditions:

(1.2.1) For any pair (X, Y) of Boolean locales in $\mathfrak{M}$, there exists at most a sole morphism from X to Y in $\mathfrak{M}$.

(1.2.2) Every trivial Boolean locale in $\mathfrak{M}$ is $\mathfrak{M}$-trivial.

(1.2.3) For any Boolean locales X, Y in $\mathfrak{M}$, if there exists a morphism from X to Y, then $Y \perp_{\mathfrak{M}} Z$ implies $X \perp_{\mathfrak{M}} Z$ for any Boolean locale Z in $\mathfrak{M}$.

(1.2.4) For any small family $\{X_\lambda\}_{\lambda \in \Lambda}$ of Boolean locales in $\mathfrak{M}$ with $X_\lambda \perp_{\mathfrak{M}} X_{\lambda'}$ for any $\lambda \neq \lambda'$, there exists a Boolean locale Z in $\mathfrak{M}$ with $Z = \Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} X_\lambda$. In particular, there exists an $\mathfrak{M}$-trivial Boolean locale in $\mathfrak{M}$.

(1.2.5) Every trivial Boolean locale in $\mathfrak{M}$ is $\mathfrak{M}$-trivial.

(1.2.6) For any Boolean locale X of the form $\Sigma_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} X_\lambda$ and any Boolean locale Y in $\mathfrak{M}$, $X_\lambda \perp_{\mathfrak{M}} Y$ for all $\lambda \in \Lambda$ implies $X \perp_{\mathfrak{M}} Y$.

(1.2.7) For any Boolean locales X and Y in $\mathfrak{M}$, $X \simeq_{\mathfrak{M}} Y$ iff there exists a Boolean locale Z in $\mathfrak{M}$ such that $X \perp_{\mathfrak{M}} Z$, $Y \perp_{\mathfrak{M}} Z$, and both of $X \oplus_{\mathfrak{M}} Z$ and $Y \oplus_{\mathfrak{M}} Z$ are $\mathfrak{M}$-maximal.

(1.2.8) For any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & Z & \end{array}$$

of **BLoc**, if f is in $\mathfrak{M}$ and h is an $\mathfrak{M}$-embedding, g is in $\mathfrak{M}$.

(1.2.9) For any object X in $\mathfrak{M}$ and any embedding f: Y → X in **BLoc**, there exists an $\mathfrak{M}$-embedding f′: Y′ → X in $\mathfrak{M}$ such that f and f′ are equivalent in **BLoc**.

Given a manual $\mathfrak{M}$ of Boolean locales, we denote by $\mathscr{L}(\mathfrak{M})$ the set $\{[X]_{\mathfrak{M}} | X \in \mathrm{Ob}\ \mathfrak{M}\}$.

Theorem 1.2.1. $\mathscr{L}(\mathfrak{M})$ is an orthocomplete orthomodular poset with respect to the following partial order and orthocomplement:

(1.2.10) $[X]_{\mathfrak{M}} \leq [Y]_{\mathfrak{M}}$ iff for any Boolean locale Z in $\mathfrak{M}$, $Y \perp_{\mathfrak{M}} Z$ implies $X \perp_{\mathfrak{M}} Z$.

(1.2.11) $[X]_{\mathfrak{M}}^{\perp} = [Y]_{\mathfrak{M}}$, where Y is a Boolean locale in $\mathfrak{M}$ such that $X \perp_{\mathfrak{M}} Y$ and $X \oplus_{\mathfrak{M}} Y$ is $\mathfrak{M}$-maximal.

For the proof of the above theorem and other details on the theory of manuals of Boolean locales, the reader is referred to Nishimura (1993b, 1995a).

Given a Boolean locale X, an X-*weight* is a function ρ_X from $\mathscr{P}(X)$ to $\overline{\mathbf{R}}^+ = \{r \in \overline{\mathbf{R}} | r \geq 0\}$ abiding by the following conditions:

(1.2.12) ρ_X is completely additive in the sense that for any disjoint family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathscr{P}(X)$ with $x = \sup_{\lambda \in \Lambda} x_\lambda$, $\rho_X(x) = \Sigma_{\lambda \in \Lambda}\ \rho_X(x_\lambda)$.

(1.2.13) ρ_X is almost finite in the sense that there exists a family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathscr{P}(X)$ with $\sup_{\lambda \in \Lambda} x_\lambda = 1_X$ and $\rho_X(x_\lambda) < +\infty$ for any $\lambda \in \Lambda$, where 1_X is the unit element of the Boolean algebra $\mathscr{P}(X)$.

An X-weight ρ_X naturally induces a Borel measure $\bar{\rho}_X$ on the Stone space Ω_X of the Boolean algebra $\mathscr{P}(X)$. Given a Borel measurable function f on Ω_X, we write $\int_X f\, d\rho_X$ for $\int_{\Omega_X} f\, d\bar{\rho}_X$ provided that the latter is meaningful in the standard sense.

Given a manual $\mathfrak{M}$ of Boolean locales, an $\mathfrak{M}$-*weight* is a family $\{\rho_X\}_{X \in \mathrm{Ob}\ \mathfrak{M}}$ of X-weights ρ_X for all $X \in \mathrm{Ob}\ \mathfrak{M}$ subject to the following condition:

(1.2.14) If $\{X_\lambda \xrightarrow{f_\lambda} X\}_{\lambda \in \Lambda}$ is an $\mathfrak{M}$-coproduct diagram in $\mathfrak{M}$, and $x \in \mathscr{P}(X)$, then $\rho_X(x) = \Sigma_{\lambda \in \Lambda}\ \rho_{X_\lambda}(\mathscr{P}(f_\lambda)(x))$.

1.3. Booleanization

Let X be a Boolean locale with $\mathbf{B} = \mathscr{P}(X)$, which shall be fixed throughout this subsection. A **B**-*valued set* is a pair of a set X and a function $[\![\cdot = \cdot]\!]_X^X: X \times X \to \mathbf{B}$ satisfying

(1.3.1) $[\![x = x']\!]_X^X = [\![x' = x]\!]_X^X$

$$(1.3.2)\quad [\![x = x']\!]^X_{\mathbb{X}} \wedge [\![x' = x'']\!]^X_{\mathbb{X}} \leq [\![x = x'']\!]^X_{\mathbb{X}}$$

for all $x, x', x'' \in X$. Unless confusion may arise, $[\![\cdot = \cdot]\!]^X_{\mathbb{X}}$ is often abbreviated to $[\![\cdot = \cdot]\!]^X$, $[\![\cdot = \cdot]\!]_{\mathbb{X}}$, or $[\![\cdot = \cdot]\!]$. The $\mathbb{X}$-set $(X, [\![\cdot = \cdot]\!]^X_{\mathbb{X}})$ is often denoted simply by X.

We denote by $\underline{\mathbf{BEns}}(\mathbb{X})$ the category of $\mathbf{B}$-valued sets, in which a morphism from $(X, [\![\cdot = \cdot]\!]^X_{\mathbb{X}})$ to $(Y, [\![\cdot = \cdot]\!]^Y_{\mathbb{X}})$ is a function $\zeta\colon X \times Y \to \mathbf{B}$ satisfying

$$(1.3.3)\quad [\![x = x']\!]^X_{\mathbb{X}} \wedge \zeta(x, y) \leq z(x', y)$$

$$(1.3.4)\quad \zeta(x, y) \wedge [\![y = y']\!]^Y_{\mathbb{X}} \leq \zeta(x, y')$$

$$(1.3.5)\quad \zeta(x, y) \wedge \zeta(x, y') \leq [\![y = y']\!]^Y_{\mathbb{X}}$$

$$(1.3.6)\quad \vee_{y \in Y}\, \zeta(x, y) = [\![x = x]\!]^X_{\mathbb{X}}$$

for all $x, x' \in X$ and $y, y' \in Y$. We denote by $\underline{\mathbf{BEns}}_0(\mathbb{X})$ the full subcategory of small$_0$ $\mathbf{B}$-valued sets of the category $\underline{\mathbf{BEns}}(\mathbb{X})$.

Given a $\mathbf{B}$-valued set $(X, [\![\cdot = \cdot]\!]^X_{\mathbb{X}})$, a function $\pi\colon X \to \mathbf{B}$ is called a *singleton* if it satisfies

$$(1.3.7)\quad \pi(x) \wedge [\![x = x']\!]^X_{\mathbb{X}} \leq \pi(x')$$

$$(1.3.8)\quad \pi(x) \wedge \pi(x') \leq [\![x = x']\!]^X_{\mathbb{X}}$$

for all $x, x' \in X$. It is easy to see that any $x \in X$ gives rise to a singleton $\{x\}$ assigning, to each $x' \in X$, $[\![x = x']\!]^X_{\mathbb{X}} \in \mathbf{B}$. The $\mathbf{B}$-valued set $(X, [\![\cdot = \cdot]\!]^X_{\mathbb{X}})$ is said to be an $\mathbb{X}$-*set* if every singleton is of the form $\{x\}$ for a unique $x \in X$. An $\mathbb{X}$-set $(Y, [\![\cdot = \cdot]\!]^Y_{\mathbb{X}})$ is said to be an $\mathbb{X}$-*subset* of another $\mathbb{X}$-set $(Z, [\![\cdot = \cdot]\!]^Z_{\mathbb{X}})$ if Y is a subset of Z and $[\![\cdot = \cdot]\!]^Y_{\mathbb{X}}$ is the restriction of $[\![\cdot = \cdot]\!]^Z_{\mathbb{X}}$. Given two $\mathbb{X}$-sets $(X_1, [\![\cdot = \cdot]\!]^{X_1}_{\mathbb{X}})$ and $(X_2, [\![\cdot = \cdot]\!]^{X_2}_{\mathbb{X}})$, their $\mathbb{X}$-*product* is $(X_1 \times_{\mathbb{X}} X_2, [\![\cdot = \cdot]\!]^{X_1 \times_{\mathbb{X}} X_2}_{\mathbb{X}})$, where:

$$(1.3.9)\quad X_1 \times_{\mathbb{X}} X_2 = \{(x_1, x_2) \mid [\![x_1 = x_1]\!]^{X_1}_{\mathbb{X}} = [\![x_2 = x_2]\!]^{X_2}_{\mathbb{X}}\}.$$

$$(1.3.10)\quad [\![(x_1, x_2) = (x_1', x_2')]\!]^{X_1 \times_{\mathbb{X}} X_2}_{\mathbb{X}} = [\![x_1 = x_1']\!]^{X_1}_{\mathbb{X}} \wedge [\![x_2 = x_2']\!]^{X_2}_{\mathbb{X}} \text{ for any } (x_1, x_2), (x_1', x_2') \in X_1 \times_{\mathbb{X}} X_2.$$

We denote by $\mathbf{BEns}(\mathbb{X})$ the full subcategory of $\mathbb{X}$-sets of the category $\underline{\mathbf{BEns}}(\mathbb{X})$, in which a morphism from an $\mathbb{X}$-set $(X, [\![\cdot = \cdot]\!]^X_{\mathbb{X}})$ to another $\mathbb{X}$-set $(Y, [\![\cdot = \cdot]\!]^Y_{\mathbb{X}})$ is known to be representable also by a function f from X to Y satisfying

$$(1.3.11)\quad [\![x = x']\!]^X_{\mathbb{X}} \leq [\![f(x) = f(x')]\!]^Y_{\mathbb{X}}$$

$$(1.3.12)\quad [\![f(x) = f(x)]\!]^Y_{\mathbb{X}} \leq [\![x = x]\!]^X_{\mathbb{X}}$$

for all $x, x' \in X$. Such a function f is called an $\mathbb{X}$-*function* from the $\mathbb{X}$-set $(X, [\![\cdot = \cdot]\!]^X_{\mathbb{X}})$ to the $\mathbb{X}$-set $(Y, [\![\cdot = \cdot]\!]^Y_{\mathbb{X}})$. As we know well, the inclusion functor $\mathbf{i}_{\mathbb{X}}\colon \mathbf{BEns}(\mathbb{X}) \to \underline{\mathbf{BEns}}(\mathbb{X})$ is an equivalence, so that it has a left adjoint functor $\mathbf{a}_{\mathbb{X}}\colon \underline{\mathbf{BEns}}(\mathbb{X}) \to \mathbf{BEns}(\mathbb{X})$. We denote by $\mathbf{BEns}_0(\mathbb{X})$ the full subcategory of small$_0$ $\mathbb{X}$-sets of the category $\mathbf{BEns}(\mathbb{X})$.

Given an X-set $(X, [\![\cdot = \cdot]\!]_X^X)$, an element x of X is said to be *total* if $[\![x = x]\!]_X^X = 1_X$, where 1_X is the unit element of the Boolean algebra **B**. The X-set $(X, [\![\cdot = \cdot]\!]_X^X)$ is said to be *total* if it has a total element.

A family $\{e_\lambda\}_{\lambda \in \Lambda}$ of elements of **B** is called a *partition of unity* of **B** if $e_\lambda \wedge e_{\lambda'} = 0_X$ for any $\lambda \neq \lambda'$, where 0_X stands for the zero element of **B**. An X-*Set* is a pair of a nonempty set X and a function $[\![\cdot = \cdot]\!]_X^X: X \times X \rightarrow$ **B** satisfying

(1.3.13) $[\![x = x]\!]_X^X = 1_X$
(1.3.14) $[\![x = x']\!]_X^X = [\![x' = x]\!]_X^X$
(1.3.15) $[\![x = x']\!]_X^X \wedge [\![x' = x'']\!]_X^X \leq [\![x = x'']\!]_X^X$

for all $x, x', x'' \in X$ and

(1.3.16) For any partition $\{e_\lambda\}_{\lambda \in \Lambda}$ of unity of **B** and a family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of X, there exists a unique $x \in X$ with $[\![x = x_\lambda]\!]_X^X \geq e_\lambda$ for all $\lambda \in \Lambda$.

We denote by **BENS**(X) the category of X-Sets, in which a morphism from an X-Set $(X, [\![\cdot = \cdot]\!]_X^X)$ to another X-Set $(Y, [\![\cdot = \cdot]\!]_X^Y)$ is a function f: $X \rightarrow Y$ satisfying

(1.3.17) $[\![x = x']\!]_X^X \leq [\![f(x) = f(x')]\!]_X^Y$

for all $x, x' \in X$.

Any total X-set $(X, [\![\cdot = \cdot]\!]_X^X)$ gives an X-Set $(\overline{X}, [\![\cdot = \cdot]\!]_X^{\overline{X}})$, where $\overline{X} = \{x \in X \mid [\![x = x]\!]_X^X = 1_X\}$ and $[\![\cdot = \cdot]\!]_X^{\overline{X}}$ is the restriction of $[\![\cdot = \cdot]\!]_X^X$ to $\overline{X}$. This yields an equivalence between the full subcategory of total X-sets of the category **BEns**(X) and the category **BENS**(X), so that henceforth the distinction between total X-sets and X-Sets will be blurred.

The category **BEns**(X) is an example of a Boolean localic topos, so that it enjoys the following *first Boolean transfer principle*.

Theorem 1.3.1. The topos **BEns**(X) enjoys all classical mathematics (=mathematics based on classical logic).

The principle plays a decisive role in Booleanization, which precedes logical quantization. A platitude in applying the principle to concepts and theorems is "Booleanize . . . ." For the detailed explanation on the above theorem, the reader is referred to Nishimura (n.d.-a). Here we content ourselves with representing the sets of complex numbers and extended real numbers within the topos **BEns**(X). Let Ω be the Stone space of the Boolean algebra **B**. Then they are represented by the set $C_{\mathbf{C}}(X)$ of complex-valued Borel functions on Ω and the set $C_{\overline{\mathbf{R}}}(X)$ of extended-real-valued Borel functions on Ω, respectively, where two functions on Ω are identified provided that they agree except for some meager Borel subset of Ω. Each element of

$\mathbf{B}$ is identified with its corresponding clopen (closed and open) subset of Ω and, at the same time, with its characteristic function, so that $\mathbf{B}$ can be put down at a subset of $C_{\mathbf{C}}(X)$ and of $C_{\overline{\mathbf{R}}}(X)$. Similarly, the sets $\mathbf{C}$ and $\overline{\mathbf{R}}$ can be reckoned to be subsets of $C_{\mathbf{C}}(X)$ and $C_{\overline{\mathbf{R}}}(X)$, respectively, by identifying each element of $\mathbf{C}$ or $\overline{\mathbf{R}}$ with its corresponding constant function on Ω. The set $C_{\mathbf{C}}(X)$ as well as $C_{\overline{\mathbf{R}}}(X)$ can be regarded as an X-Set with respect to:

(1.3.18) $[\![\alpha = \beta]\!]_X = \sup\{e \in \mathbf{B} \mid e\alpha = e\beta\}$ for all α, β of $C_{\mathbf{C}}(X)$ or of $C_{\overline{\mathbf{R}}}(X)$, respectively.

The X-Sets $C_{\mathbf{C}}(X)$ and $C_{\overline{\mathbf{R}}}(X)$ inherit their algebraic structures from the set $\mathbf{C}$ of complex numbers and the set $\overline{\mathbf{R}}$ of extended real numbers, respectively. By way of example, the (complex) conjugation in $\mathbf{C}$ gives rise to the pointwise conjugation in $C_{\mathbf{C}}(X)$, namely,

(1.3.19) $\overline{\alpha}(\omega) = \overline{\alpha(\omega)}$ for any $\alpha \in C_{\mathbf{C}}(X)$ and $\omega \in \Omega$.

By simply interpreting the notion of a small category within the topos $\mathbf{BEns}(X)$, we get the notion of an X-*category*, which is externally a six-tuple $\mathscr{C} = (\mathrm{Ob}\,\mathscr{C}, \mathrm{Mor}\,\mathscr{C}, d_{\mathscr{C}}, r_{\mathscr{C}}, \mathrm{id}_{\mathscr{C}}, \circ_{\mathscr{C}})$, where:

(1.3.20) $\mathrm{Ob}\,\mathscr{C}$ and $\mathrm{Mor}\,\mathscr{C}$ are X-sets.
(1.3.21) $d_{\mathscr{C}}$ and $r_{\mathscr{C}}$ are X-functions from $\mathrm{Mor}\,\mathscr{C}$ to $\mathrm{Ob}\,\mathscr{C}$.
(1.3.22) $\mathrm{id}_{\mathscr{C}}$ is an X-function from $\mathrm{Ob}\,\mathscr{C}$ to $\mathrm{Mor}\,\mathscr{C}$ such that $[\![x = y]\!]_X^{\mathrm{Ob}\,\mathscr{C}} = [\![\mathrm{id}_{\mathscr{C}}(x) = \mathrm{id}_{\mathscr{C}}(y)]\!]_X^{\mathrm{Mor}\,\mathscr{C}}$ for all $x, y \in \mathrm{Ob}\,\mathscr{C}$.
(1.3.23) $\circ_{\mathscr{C}}$ is an X-function from $\mathrm{Mor}\,\mathscr{C} \times_{\mathrm{Ob}\,\mathscr{C}} \mathrm{Mor}\,\mathscr{C}$ to $\mathrm{Mor}\,\mathscr{C}$.
(1.3.24) If we regard $\mathrm{Ob}\,\mathscr{C}$ and $\mathrm{Mor}\,\mathscr{C}$ as mere sets, then the six-tuple $(\mathrm{Ob}\,\mathscr{C}, \mathrm{Mor}\,\mathscr{C}, d_{\mathscr{C}}, r_{\mathscr{C}}, \mathrm{id}_{\mathscr{C}}, \circ_{\mathscr{C}})$ is a category in the usual sense.

By way of example, the totality of $\mathbf{BEns}_0(X_e)$'s for all $e \in \mathbf{B}$ naturally forms an X-category to be denoted by $\mathscr{B}\mathscr{E}\mathscr{n}\mathscr{s}_0(X)$, as explained in detail in Nishimura (1995b, Example 1.1).

By interpreting the notion of a functor of small categories within the topos $\mathbf{BEns}(X)$, we get the notion of an X-functor from an X-category $\mathscr{C}$ to an X-category $\mathscr{D}$, which is a functor from the category $\mathscr{C}$ to the category $\mathscr{D}$ satisfying the following condition:

(1.3.25) The assignment $f \in \mathrm{Mor}\,\mathscr{C} \mapsto \mathscr{F}(f) \in \mathrm{Mor}\,\mathscr{D}$ is an X-function.

1.4. Relations Between Two Booleanizations

Let $f\colon X_- \to X_+$ be a morphism of Boolean locales, which shall be fixed throughout this subsection. Let $\mathbf{B}_\pm = \mathscr{P}(X_\pm)$ and let $\Omega_\pm$ be the Stone spaces of $\mathbf{B}_\pm$. Given an X_+-set $(X, [\![\cdot = \cdot]\!]_{X_+}^X)$, the pair $(X, \mathscr{P}(f)([\![\cdot = \cdot]\!]_{X_+}^X))$ is a $\mathbf{B}_-$-valued set, which gives rise, after the application of the functor $\mathbf{a}_{X_-}$: $\underline{\mathbf{BEns}}(X_-) \to \mathbf{BEns}(X_-)$, to an X_--set $f^*(X, [\![\cdot = \cdot]\!]_{X_+}^X)$. The function f^*

from Ob $\mathbf{BEns}(X_+)$ to Ob $\mathbf{BEns}(X_-)$ can readily be extended to a functor from $\mathbf{BEns}(X_+)$ to $\mathbf{BEns}(X_-)$, which we denote by the same symbol f^*.

The following is the *second Boolean transfer principle*, which plays a decisive role in connecting two Booleanizations (over distinct Boolean locales).

Theorem 1.4.1. The functor $\mathbf{f}^*$: $\mathbf{BEns}(X_+) \to \mathbf{BEns}(X_-)$ preserves every (many-sorted) first-order structure.

By way of example, f^* induces a functor f^*_{Grp} from the category $\mathbf{Grp}(X_+)$ of groups within the topos $\mathbf{BEns}(X_+)$ to the category $\mathbf{Grp}(X_-)$ of groups within the topos $\mathbf{BEns}(X_-)$. For the proof and the exact meaning of the above theorem, see MacLane and Moerdijk (1992, Chapter IX, §7, Proposition 2, and Chapter X, §3, Corollary 4).

The complete Boolean homomorphism $\mathscr{P}(f)$: $\mathbf{B}_+ \to \mathbf{B}_-$ induces a continuous mapping $\mathscr{S}(f)$: $\Omega_- \to \Omega_+$ by the so-called Stone duality. This gives rise to a function $\alpha \in C_{\mathbf{C}}(X_+) \mapsto \alpha \circ \mathscr{S}(f) \in C_{\mathbf{C}}(X_-)$, which we denote by $f^\#_{\mathbf{C}}$.

Given $X_\pm$-sets $(X_\pm, [\![\cdot = \cdot]\!]^{X_\pm}_{X_\pm})$, a function f: $X_+ \to X_-$ is called an f-*function* from the X_+-set $(X_+, [\![\cdot = \cdot]\!]^{X_+}_{X_+})$ to the X_--set $(X_-, [\![\cdot = \cdot]\!]^{X_-}_{X_-})$ if it satisfies

$$(1.4.1) \quad \mathscr{P}(f)([\![x = x']\!]^{X_+}_{X_+}) \leq [\![f(x) = f(x')]\!]^{X_-}_{X_-}$$
$$(1.4.2) \quad [\![f(x) = f(x)]\!]^{X_-}_{X_-} \leq \mathscr{P}(f)([\![x = x]\!]^{X_+}_{X_+})$$

foı all $x, x' \in X_+$. Since the functor $\mathbf{a}_{X_-}$ is left adjoint to the functor $\mathbf{i}_{X_-}$, we have the following result.

Proposition 1.4.2. There is a bijective correspondence between the f-functions from $(X_+, [\![\cdot = \cdot]\!]^{X_+}_{X_+})$ to $(X_-, [\![\cdot = \cdot]\!]^{X_-}_{X_-})$ and the X_--functions from $f^*(X_+, [\![\cdot = \cdot]\!]^{X_+}_{X_+})$ to $(X_-, [\![\cdot = \cdot]\!]^{X_-}_{X_-})$.

Given $X_\pm$-categories $\mathscr{C}_\pm$, a functor $\mathscr{F}$ from the category $\mathscr{C}_+$ to the category $\mathscr{C}_-$ is called an f-*functor* if it yields the following condition:

(1.4.3) The assignment $f \in \mathrm{Mor}\,\mathscr{C}_+ \mapsto \mathscr{F}(f) \in \mathrm{Mor}\,\mathscr{C}_-$ is an f-function.

1.5. Quantization

Let us introduce the category to be denoted by **BCat**. Its objects are all pairs $(X, \mathscr{A})$ of a Boolean locale X and an X-category $\mathscr{A}$. A morphism from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$ in **BCat** is a pair $(f, \mathscr{F})$ of a morphism f: $X \to Y$ in **BLoc** and an f-functor $\mathscr{F}: \mathscr{B} \to \mathscr{A}$. The composition $(g, \mathscr{G}) \circ (f, \mathscr{F})$ of morphisms $(f, \mathscr{F})$: $(X, \mathscr{A}) \to (Y, \mathscr{B})$ and $(g, \mathscr{G})$: $(Y, \mathscr{B}) \to (Z, \mathscr{C})$ in **BCat** is defined to be $(g \circ f, \mathscr{F} \circ \mathscr{G})$. As discussed in Nishimura (1995b, §3), the category **BCat** has small coproducts. The assignments $(X, \mathscr{A}) \in$ Ob **BCat** $\mapsto X \in$ Ob **BLoc** and $(f, \mathscr{F}) \in$ Mor **BCat** $\mapsto$ $f \in$ Mor **BLoc** constitute a functor to be denoted by $\Theta_{\mathbf{BLoc}}$.

We now introduce a category to be denoted by **BObj**. Its objects are all triples $(X, \mathscr{A}, a)$ such that $(X, \mathscr{A}) \in$ Ob **BCat** and a is a total object of the X-category $\mathscr{A}$. A morphism from $(X, \mathscr{A}, a)$ to $(Y, \mathscr{B}, b)$ in **BObj** is a triple $(f, \mathscr{F}, \mathscr{f})$ such that $(f, \mathscr{F})$ is a morphism from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$ in the category **BCat** and $\mathscr{f}$ is a total morphism from $\mathscr{F}b$ to a in the X-category $\mathscr{A}$. The composition $(g, \mathscr{G}, \mathscr{g}) \circ (f, \mathscr{F}, \mathscr{f})$ of $(f, \mathscr{F}, \mathscr{f})$: $(X, \mathscr{A}, a) \to (Y, \mathscr{B}, b)$ and $(g, \mathscr{G}, \mathscr{g})$: $(Y, \mathscr{B}, b) \to (Z, \mathscr{C}, c)$ in **BObj** is defined to be $(g \circ f, \mathscr{F} \circ \mathscr{G}, \mathscr{f} \circ \mathscr{F}_{\mathscr{g}})$. It is easy to see that the category **BObj** has small coproducts. The assignments

$$(X, \mathscr{A}, a) \in \text{Ob } \mathbf{BObj} \mapsto (X, \mathscr{A}) \in \text{Ob } \mathbf{BCat}$$
$$(f, \mathscr{F}, \mathscr{f}) \in \text{Ob } \mathbf{BObj} \mapsto (f, \mathscr{F}) \in \text{Mor } \mathbf{BCat}$$

constitute a functor from the category **BObj** to the category **BCat** to be denoted by $\Theta_{\mathbf{BCat}}$.

Let $\mathfrak{M}$ be a manual of Boolean locales, which shall be fixed throughout the rest of this subsection. An *empirical framework over* $\mathfrak{M}$ is a functor Φ from $\mathfrak{M}$ to **BCat** subject to the following conditions:

(1.5.1) It maps $\mathfrak{M}$-coproduct diagrams to coproduct diagrams in **BCat**.
(1.5.2) $\Theta_{\mathbf{BLoc}} \circ \Phi$ is the identity functor of $\mathfrak{M}$ into **BLoc**.

For an empirical framework Φ over $\mathfrak{M}$, we denote by $\Phi_{\mathscr{C}at}$ the function with the same domain of Φ such that $\Phi(X) = (X, \Phi_{\mathscr{C}at}(X))$ for each $X \in$ Ob $\mathfrak{M}$ and $\Phi(f) = (f, \Phi_{\mathscr{C}at}(f))$ for each $f \in$ Mor $\mathfrak{M}$.

Given an empirical framework Φ over $\mathfrak{M}$, we now introduce a category to be denoted by **EObj**(Φ). Its objects are all functors $\mathfrak{F}$ from $\mathfrak{M}$ to **BObj** abiding by the following conditions:

(1.5.3) It maps $\mathfrak{M}$-coproduct diagrams in $\mathfrak{M}$ to coproduct diagrams in **BObj**.
(1.5.4) $\Theta_{\mathbf{BCat}} \circ \mathfrak{F} = \Phi$.

Given such a functor $\mathfrak{F}$: $\mathfrak{M} \to$ **BObj**, we denote by $\mathfrak{F}_{\mathscr{O}bj}$ the function with the same domain of $\mathfrak{F}$ such that the value of $\mathfrak{F}_{\mathscr{O}bj}(\cdot)$ is the third component of the triple $\mathfrak{F}(\cdot)$. A morphism from $\mathfrak{F}$ to $\mathfrak{G}$ in **EObj**(Φ) is an assignment ζ to each $X \in$ Ob $\mathfrak{M}$ of a total morphism ζ_X: $\mathfrak{F}_{\mathscr{O}bj}(X) \to \mathfrak{F}_{\mathscr{O}bj}(Y)$ satisfying the following condition:

(1.5.5) The diagram

$$\begin{array}{ccc} \Phi_{\mathcal{C}at}(f)(\mathfrak{F}_{\mathcal{O}bj}(Y)) & \xrightarrow{\mathfrak{F}_{\mathcal{O}bj}(f)} & \mathfrak{F}_{\mathcal{O}bj}(X) \\ \Big\downarrow \Phi_{\mathcal{C}at}(f)(\zeta_Y) & & \Big\downarrow \zeta_X \\ \Phi_{\mathcal{C}at}(f)(\mathfrak{G}_{\mathcal{O}bj}(Y)) & \xrightarrow[\mathfrak{G}_{\mathcal{O}bj}(f)]{} & \mathfrak{G}_{\mathcal{O}bj}(X) \end{array}$$

is commutative for every $f: X \to Y \in \text{Mor } \mathfrak{M}$.

The composition $\eta \circ \zeta$ of morphisms $\zeta: \mathfrak{F} \to \mathfrak{G}$ and $\eta: \mathfrak{G} \to \mathfrak{H}$ in **EObj**(Φ) is defined to be the assignment $X \in \text{Ob } \mathfrak{M} \mapsto \eta_X \circ \zeta_X$.

1.6. Hilbert Space Theory

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, which shall be fixed throughout this subsection. The main purpose of this subsection is to review some celebrated theorems, which will be Booleanized and then made empirical in later sections. Let us begin with the so-called square root lemma.

Theorem 1.6.1. For any bounded positive operator A on H, there exists a unique bounded positive operator B on H with $A = B^2$. Furthermore, B commutes with every bounded operator on H which commutes with A. This B is denoted by $A^{1/2}$.

Proof. See Theorem VI.9 of Reed and Simon (1972). ■

Recall that a bounded operator T on H is called:

(1.6.1) *Self-adjoint* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$.
(1.6.2) *Positive* if $\langle Tx, x \rangle \geq 0$ for all $x \in H$.

It is well known that a bounded positive operator is self-adjoint. A *proposition operator* on H is a idempotent bounded self-adjoint operator. A *unitary operator* on H is a Hilbert map of H onto itself.

Proposition 1.6.2. If a Hilbert subspace K of H is invariant under a bounded self-adjoint operator A, then the orthogonal complement $K^{\perp}$ of K in H is also invariant under A.

Proof. Let P_K be the projection operator corresponding to K and I the identity operator on H. Then the invariance of K under A is equivalent to

$P_KAP_K = AP_K$, which implies $P_KAP_K = (P_KAP_K)^* = (AP_K)^* = P_KA$. Therefore $AP_K = P_KA$. This implies

$$(I - P_K)A(I - P_K) = A - AP_K + P_KA - P_KAP_K = A - AP_K$$
$$= A(I - P_K)$$

which is tantamount to the invariance of $K^{\perp}$ under A. ■

Corollary 1.6.3. If a Hilbert subspace K of H is invariant under a bounded self-adjoint operator A, then it is also invariant under $A^{1/2}$.

Proof. Since K is invariant under A, A commutes with P_K by the above proposition, which implies by Theorem 1.6.1 that $A^{1/2}$ commutes with P_K. Therefore K is invariant under $A^{1/2}$. ■

A Boolean version of the following simple proposition will be useful in subsequent sections.

Proposition 1.6.4. Let H' and K be Hilbert subspaces of H and K' a Hilbert subspace of K. Let $P_{H'}$ be the projection operator of H onto H' and $P_{K'}$ the projection operator of K onto K'. Then $P_{K'}x = P_{H'}x$ for all $x \in K$ iff $K' \subseteq H'$ and $K'^{\perp} \subseteq H'^{\perp}$, where $K'^{\perp}$ denotes the orthogonal complement of K' in K and $H'^{\perp}$ denotes the orthogonal complement of H' in H.

Recall that a (possibly unbounded) operator T on H is a linear mapping from a linear subspace $\mathscr{D}(T)$ of H into H. It is called *self-adjoint* if the orthogonal complement of the graph $\Gamma(T) = \{(x, Tx) \mid x \in \mathscr{D}(T)\}$ of T in $H \oplus H$ is $\{(-Tx, x) \mid x \in \mathscr{D}(T)\}$. It is easy to see that a bounded operator is self-adjoint in the earlier sense iff it is so in the present sense. It is also easy to see that every self-adjoint operator is a closed, densely defined operator.

We denote by $\mathscr{B}_0(\mathbf{R})$ the field of subsets of $\mathbf{R}$ generated by $(-\infty, a) = \{r \in \mathbf{R} \mid r < a\}$ and $(-\infty, a] = \{r \in \mathbf{R} \mid r \leq a\}$ for all $a \in \mathbf{R}$. The σ-field generated by $\mathscr{B}_0(\mathbf{R})$ is denoted by $\mathscr{B}(\mathbf{R})$. A *spectral measure* is a function E on $\mathscr{B}_0(\mathbf{R})$ subject to the following conditions:

(1.6.3) $E(M)$ is a projection operator for each $M \in \mathscr{B}_0(\mathbf{R})$.

(1.6.4) $E(\mathbf{R}) = I$.

(1.6.5) $E(M \cup N) = E(M) + E(N)$ for any disjoint sets M and N in $\mathscr{B}_0(\mathbf{R})$.

(1.6.6) If $\{M_n\}_{n\in\mathbf{N}}$ is an increasing sequence of sets in $\mathscr{B}_0(\mathbf{R})$ such that $\cup_{n\in\mathbf{N}} M_n$ is also in $\mathscr{B}_0(\mathbf{R})$, then $E(\cup_{n\in\mathbf{N}} M_n) = \mathrm{LUB}_{n\in\mathbf{N}} E(M_n)$, where $\mathrm{LUB}_{n\in\mathbf{N}} E(M_n)$ stands for the least upper bound of $E(M_n)$'s within the lattice of all projection operators.

Given a spectral measure E, each $x \in H$ determines a measure $M \in \mathscr{B}_0(\mathbf{R}) \mapsto \langle x, E(M)x\rangle$, which can be extended uniquely to a measure on $\mathscr{B}(\mathbf{R})$ usually to be denoted by $d\langle x, E_\lambda x\rangle$.

The celebrated spectral theorem for self-adjoint operators goes as follows:

Theorem 1.6.5. Each spectral measure E determines a unique self-adjoint operator on H, usually denoted by $\int_{-\infty}^{+\infty}\lambda\ dE_\lambda$, subject to the following conditions:

(1.6.7) $\mathscr{D}(\int_{-\infty}^{+\infty}\lambda\ dE_\lambda) = \{x \in H \mid \int_{-\infty}^{+\infty}\lambda^2\ d\langle x, E_\lambda x\rangle\}$.
(1.6.8) $\langle x, (\int_{-\infty}^{+\infty}\lambda\ dE_\lambda)x\rangle = \int_{-\infty}^{+\infty}\lambda\ d\langle x, E_\lambda x\rangle$ for any $x \in \mathscr{D}(\int_{-\infty}^{+\infty}\lambda\ dE_\lambda)$.

The above mapping $E \mapsto \int_{-\infty}^{+\infty}\lambda\ dE_\lambda$ gives rise to a bijective correspondence between the spectral measures and the self-adjoint operators on H.

A mapping assigning, to each $t \in \mathbf{R}$, a unitary operator $U(t)$ on H is called a *one-parameter unitary group* on H if it satisfies the following conditions:

(1.6.9) $U(s + t) = U(s)U(t)$ for all $s, t \in \mathbf{R}$.
(1.6.10) The mapping $t \in \mathbf{R} \mapsto U(t)$ is strongly continuous. I.e., for any $x \in H$ and any $t_0 \in \mathbf{R}$, $t \to t_0$ implies $U(t)x \to U(t_0)x$.

The following two theorems are well known as Stone's theorem.

Theorem 1.6.6. Let A be a (possibly unbounded) self-adjoint operator on H. Then the mapping $t \in \mathbf{R} \mapsto \exp(itA)$ is a one-parameter unitary group on H.

Theorem 1.6.7. Let $t \in \mathbf{R} \mapsto U(t)$ be a one-parameter unitary group on H. Then there exists a self-adjoint operator A on H with $U(t) = \exp(itA)$ for all $t \in \mathbf{R}$. For any $x \in H$, x belongs to the domain $\mathscr{D}(A)$ of A iff $\lim_{t\to 0}\{[U(t)x - x]/t\}$ exists, in which we have $Ax = i^{-1}\lim_{t\to 0}\{[U(t)x - x]/t\}$.

2. EMPIRICAL HILBERT SPACES

2.1. Booleanization

Let X be a Boolean locale, which shall be fixed throughout this subsection. Let $\mathbf{B} = \mathscr{P}(\mathrm{X})$. By interpreting the notion of a pre-Hilbert space in the topos **BEns**(X), we get the notion of an X-*pre-Hilbert space*, which can be defined as a $C_\mathrm{C}(\mathrm{X})$-module $\mathscr{H}$ endowed with a function $\langle \cdot, \cdot \rangle_{\mathscr{H}}: \mathscr{H} \times \mathscr{H} \to C_\mathrm{C}(\mathrm{X})$ subject to the following conditions:

(2.1.1) For any family $\{e_\lambda\}_{\lambda\in\Lambda}$ of elements in $\mathbf{B}$ and any $x, y \in C_\mathrm{C}(\mathrm{X})$, if $e_\lambda x = e_\lambda y$ for any $\lambda \in \Lambda$, then $(\sup_{\lambda\in\Lambda} e_\lambda)x = (\sup_{\lambda\in\Lambda} e_\lambda)y$.
(2.1.2) For any family $\{x_\lambda\}_{\lambda\in\Lambda}$ of elements of $\mathscr{H}$ and any partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity in $\mathbf{B}$, there exists a unique $x \in \mathscr{H}$ such that $e_\lambda x = e_\lambda x_\lambda$ for any $\lambda \in \Lambda$.

(2.1.3) $\langle \alpha x_1 + \beta x_2, y\rangle_{\mathscr{H}} = \alpha\langle x_1, y\rangle_{\mathscr{H}} + \beta\langle x_2, y\rangle_{\mathscr{H}}$ for any $\alpha, \beta \in C_{\mathrm{C}}(\mathrm{X})$ and any $x_1, x_2, y \in \mathscr{H}$.
(2.1.4) $\overline{\langle x, y\rangle_{\mathscr{H}}} = \langle y, x\rangle_{\mathscr{H}}$ for any $x, y \in \mathscr{H}$.
(2.1.5) $\langle x, x\rangle_{\mathscr{H}} \geq 0$ for any $x \in \mathscr{H}$, and the equality holds only if $x = 0$.

The function $\langle\cdot, \cdot\rangle_{\mathscr{H}}$ is often denoted simply by $\langle\cdot, \cdot\rangle$ unless confusion may arise. The notation $\|x\|$ stands for $\langle x, x\rangle^{1/2}$. The X-pre-Hilbert space is usually denoted simply by $\mathscr{H}$ rather than exactly by $(\mathscr{H}, \langle\cdot, \cdot\rangle_{\mathscr{H}})$. The pre-Hilbert space $\mathscr{H}$ can naturally be made an X-Set provided it is endowed with the function $[\![\cdot = \cdot]\!]^{\mathscr{H}}_{\mathrm{X}}: \mathscr{H} \times \mathscr{H} \to \mathbf{B}$ to be defined as follows:

(2.1.6) $[\![x = y]\!]^{\mathscr{H}}_{\mathrm{X}} = \sup\{e \in \mathscr{P}(\mathrm{X}) \mid ex = ey\}$ for $x, y \in \mathscr{H}$.

Proposition 2.1.1. $(\mathscr{H}, [\![\cdot = \cdot]\!]^{\mathscr{H}}_{\mathrm{X}})$ is an X-Set.

Corollary 2.1.2. The action of $C_{\mathrm{C}}(\mathrm{X})$ on $\mathscr{H}$ is an X-function from $C_{\mathrm{C}}(\mathrm{X}) \times_{\mathrm{X}} \mathscr{H}$ to $\mathscr{H}$.

A sequence $\{x_n\}_{n\in\mathbf{N}}$ in an X-pre-Hilbert space $\mathscr{H}$ is said to *X-converge to* an element x of $\mathscr{H}$, in notation X-$\lim_{n\to\infty} x_n = x$, if for any $\epsilon > 0$, there exist a partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity in $\mathbf{B}$ and a family $\{n_\lambda\}_{\lambda\in\Lambda}$ of natural numbers satisfying the following condition:

(2.1.7) For any $\lambda \in \Lambda$ and any natural number $n > n_\lambda$, $\|e_\lambda x_n - e_\lambda x\| < \epsilon$.

The above notion of X-convergence can be generalized in various ways. By way of example, for a family $\{x_\alpha\}_{\alpha\in\mathbf{R}^\times}$ of elements of an X-pre-Hilbert space $\mathscr{H}$ indexed by $\mathbf{R}^\times = \{t \in \mathbf{R} \mid t \neq 0\}$, we write X-$\lim_{t\to 0} x_t = x_0$ with $x_0 \in \mathscr{H}$ if for any $\epsilon > 0$, there exist a partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity of $\mathbf{B}$ and a family $\{\delta_\lambda\}_{\lambda\in\Lambda}$ of positive numbers satisfying the following condition:

(2.1.8) For any $\lambda \in \Lambda$ and any $t \in \mathbf{R}^\times$ with $|t| < \delta_\lambda$, $\|e_\lambda x_t - e_\lambda x\| < \epsilon$.

Let $\mathscr{H}$ be an X-pre-Hilbert space. A sequence $\{x_n\}_{n\in\mathbf{N}}$ in $\mathscr{H}$ is said to be X-*Cauchy* if for any $\epsilon > 0$, there exist a partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity in $\mathbf{B}$ and a family $\{n_\lambda\}_{\lambda\in\Lambda}$ of natural numbers satisfying the following condition:

(2.1.9) For any $\lambda \in \Lambda$ and any natural numbers m, n larger than n_λ, $\|e_\lambda x_m - e_\lambda x_n\| < \epsilon$.

An X-pre-Hilbert space $\mathscr{H}$ is called an X-*Hilbert space* if any X-Cauchy sequence $\{x_n\}_{n\in\mathbf{N}}$ X-converges to an element x in $\mathscr{H}$. This is the interpretation of the notion of a Hilbert space in the topos **BEns**(X). For another formulation of a X-Hilbert space, see Ozawa (1984), who called it an AW^*-module. If **B** happens to be the projection lattice of an Abelian von Neumann algebra, the notion can be defined as the normal module over the von Neumann algebra, as Ozawa (1983) did.

An X-subset $\mathscr{X}$ of an X-pre-Hilbert space $\mathscr{H}$ is said to be *X-dense* in $\mathscr{H}$ if for any $x \in \mathscr{H}$ there exists a sequence $\{x_n\}_{n\in \mathrm{N}}$ of elements of $\mathscr{X}$ with X-$\lim_{n\to\infty} x_n = x$. An X-Hilbert space $\tilde{\mathscr{H}}$ is said to be an *X-completion* of an X-pre-Hilbert space $\mathscr{H}$ if $\mathscr{H}$ is X-dense in $\tilde{\mathscr{H}}$. By using the first Boolean transfer principle, we can deduce from the unique existence of a completion of a pre-Hilbert space that there exists an essentially unique X-completion of a given X-pre-Hilbert space.

By interpreting the notion of a Hilbert subspace of a Hilbert space within the topos **BEns**(X), we get the notion of an *X-Hilbert subspace* of an X-Hilbert space $\mathscr{H}$. It is defined as a **C**(X)-submodule $\mathscr{H}'$ of $\mathscr{H}$ which is an X-Hilbert space with respect to the restriction of $\langle \cdot, \cdot\rangle_{\mathscr{H}}$ to $\mathscr{H}'$. The *X-orthogonal complement* of $\mathscr{H}'$ in $\mathscr{H}$, denoted by $\mathscr{H}'^{\perp}$, is defined to be $\{x \in \mathscr{H} \mid \langle x, y\rangle_{\mathscr{H}} = 0$ for any $y \in \mathscr{H}'\}$, which is easily seen to be an X-Hilbert subspace of $\mathscr{H}$.

By interpreting the notion of a linear map of Hilbert spaces in the topos **BEns**(X), we get the notion of an *X-linear map* from an X-Hilbert space $\mathscr{H}_1$ to an X-Hilbert space $\mathscr{H}_2$, which is a function φ from $\mathscr{H}_1$ to $\mathscr{H}_2$ subject to the following conditions:

(2.1.10) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for any $x, y \in \mathscr{H}_1$.
(2.1.11) $\varphi(\alpha x) = \alpha\varphi(x)$ for any $\alpha \in C_{\mathbf{C}}(\mathrm{X})$ and any $x \in \mathscr{H}_1$.

By interpreting the notion of a Hilbert map of Hilbert spaces in the topos **BEns**(X), we get the notion of an *X-Hilbert map* from an X-Hilbert space $\mathscr{H}_1$ to an X-Hilbert space $\mathscr{H}_2$, which is an X-linear map from $\mathscr{H}_1$ to $\mathscr{H}_2$ subject to the following condition:

(2.1.12) $\langle\varphi(x), \varphi(y)\rangle_{\mathscr{H}_2} = \langle x, y\rangle_{\mathscr{H}_1}$ for any $x, y \in \mathscr{H}_1$.

We denote by **BHil**(X) the category of X-Hilbert spaces and X-Hilbert maps, whose full subcategory of small_0 X-Hilbert spaces is denoted by $\mathbf{BHil}_0$(X). Just as the totality of $\mathbf{BEns}_0(\mathrm{X}_e)$'s ($e \in \mathbf{B}$) forms an X-category $\mathscr{BE}ns_0$(X), the totality of $\mathbf{BHil}_0(\mathrm{X}_e)$'s ($e \in \mathbf{B}$) forms an X-category $\mathscr{BH}il$(X).

2.2. Relations Between Two Booleanizations

Let f: $\mathrm{X}_- \to \mathrm{X}_+$ be a morphism of Boolean locales, which shall be fixed throughout this subsection. Given $\mathrm{X}_\pm$-Hilbert spaces $\mathscr{H}_\pm$, an *f-linear map* from $\mathscr{H}_+$ to $\mathscr{H}_-$ is a function φ from $\mathscr{H}_+$ to $\mathscr{H}_-$ subject to the following conditions:

(2.2.1) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for any $x, y \in \mathscr{H}_+$.
(2.2.2) $\varphi(\alpha x) = \mathfrak{f}^{\#}_{\mathbf{C}}(\alpha)\varphi(x)$ for any $\alpha \in C_{\mathbf{C}}(\mathrm{X}_+)$ and any $x \in \mathscr{H}_+$.

It is easy to see the following result.

Proposition 2.2.1. An f-linear map from a Hilbert X_+-space $\mathscr{H}_+$ to a Hilbert space $\mathscr{H}_-$ is an f-function from the X_+-set $\mathscr{H}_+$ to the X_--set $\mathscr{H}_-$, where $\mathscr{H}_\pm$ are regarded as $X_\pm$-sets in the sense of Proposition 2.1.1.

An f-linear map φ from an X_+-Hilbert space $\mathscr{H}_+$ to an X_--Hilbert space $\mathscr{H}_-$ is called an *f-Hilbert map* from $\mathscr{H}_+$ to $\mathscr{H}_-$ if φ satisfies the following condition:

(2.2.3) $\langle\varphi(x), \varphi(y)\rangle_{\mathscr{H}_-} = f^{\#}_{\mathbb{C}}(\langle x, y\rangle_{\mathscr{H}_+})$ for any $x, y \in \mathscr{H}_+$.

Given an X_+-Hilbert space $\mathscr{H}_+$, the second Boolean transfer principle guarantees that $f^*\mathscr{H}_+$ is an X_--pre-Hilbert space, whose X_--completion is also denoted by $f^*\mathscr{H}_+$. Henceforth $f^*\mathscr{H}_+$ shall denote the latter entity unless stated to the contrary. The canonical f-Hilbert map from $\mathscr{H}_+$ to $f^*\mathscr{H}_+$ is denoted by $\eta_{\mathscr{H}_+,f}$.

As in Proposition 1.4.2, we have the following result.

Proposition 2.2.2. Given $X_\pm$-Hilbert spaces $\mathscr{H}_\pm$, the assignment of $\varphi \circ \eta_{\mathscr{H}_+,f}$ to each Hilbert X_--map φ: $f^*\mathscr{H}_+ \to \mathscr{H}_-$ gives a bijection between the X_--Hilbert maps from $f^*\mathscr{H}_+$ to $\mathscr{H}_-$ and the f-Hilbert maps from $\mathscr{H}_+$ to $\mathscr{H}_-$.

Proof. Booleanize the so-called B.L.T. Theorem (Reed and Simon, 1972, Theorem I.7). ■

Let $\mathscr{H}_\pm$ be $X_\pm$-Hilbert spaces. Given an X_--Hilbert map φ: $f^*\mathscr{H}_+ \to \mathscr{H}_-$, we denote $\varphi \circ \eta_{\mathscr{H}_+,f}$ by $\varphi^\vee$. Given an f-Hilbert map ψ: $\mathscr{H}_+ \to \mathscr{H}_-$, we denote by $\psi^\wedge$ the X_--Hilbert map from $f^*\mathscr{H}_+$ to $\mathscr{H}_-$ with $\psi = \psi^\wedge \circ \eta_{\mathscr{H}_+,f}$.

The assignment $\mathscr{H}_+ \in$ Ob $\mathbf{BHil}_0(X_+) \mapsto f^*\mathscr{H}_+ \in$ Ob $\mathbf{BHil}_0(X_-)$ can be extended readily to a functor from the category $\mathbf{BHil}_0(X_+)$ to the category $\mathbf{BHil}_0(X_-)$, which we denote by $f^*_{\mathbf{BHil}_0}$. The totality of $f[e]^*_{\mathbf{BHil}_0}$'s for all $e \in \mathscr{P}(X_+)$ forms an f-functor from the X_+-category $\mathscr{BHil}_0(X_+)$ to the X_--category $\mathscr{BHil}_0(X_-)$, which we denote by $f^*_{\mathscr{BHil}_0}$.

2.3. Quantization

Let $\mathfrak{M}$ be a small$_0$ manual of Boolean locales, which shall be fixed throughout this subsection. We denote by $\mathfrak{EHil}_0(\mathfrak{M})$ the empirical framework over $\mathfrak{M}$ consisting of assignments $X \in$ Ob $\mathfrak{M} \mapsto (X, \mathscr{BHil}_0(X))$ and

$$f: X_- \to X_+ \in \text{Mor } \mathfrak{M} \mapsto (f, f^*_{\mathscr{BHil}_0}): (X_-, \mathscr{BHil}_0(X_-)) \to (X_+, \mathscr{BHil}_0(X_+))$$

An *empirical Hilbert space over* $\mathfrak{M}$ or simply an $\mathfrak{M}$*-Hilbert* $\mathfrak{M}$*-space* is a map $\mathfrak{H}$ assigning of a small$_0$ X-Hilbert space $\mathfrak{H}(X)$ to each $X \in$ Ob $\mathfrak{M}$ and of an f-Hilbert map $\mathfrak{H}(f)$: $\mathfrak{H}(X_+) \to \mathfrak{H}(X_-)$ to each f: $X_- \to X_+ \in$ Mor $\mathfrak{M}$ subject to the following condition:

(2.3.1) The assignments $X \in \text{Ob } \mathfrak{M} \mapsto (X, \mathscr{BH}\ell_0(X), \mathfrak{H}(X))$ and f: $X_- \to X_+ \in \text{Mor } \mathfrak{M} \mapsto (f, f^*_{\mathscr{BH}\ell_0}, \mathfrak{H}(f)\hat{})$ constitute an object of $\mathbf{EObj}(\mathfrak{E}\mathfrak{H}\mathfrak{il}_0(\mathfrak{M}))$.

An $\mathfrak{M}$-Hilbert $\mathfrak{M}$-space $\mathfrak{H}'$ is said to be an *$\mathfrak{M}$-Hilbert $\mathfrak{M}$-subspace* of an $\mathfrak{M}$-Hilbert $\mathfrak{M}$-space $\mathfrak{H}$ if it satisfies the following conditions:

(2.3.2) For any $X \in$ Ob $\mathfrak{M}$, $\mathfrak{H}'(X)$ is an X-Hilbert subspace of $\mathfrak{H}(X)$.

(2.3.3) For any f: $X_- \to X_+ \in$ Mor $\mathfrak{M}$, we have $\mathfrak{H}(f)(\mathfrak{H}'(X_+)) \subset \mathfrak{H}'(X_-)$ and $\mathfrak{H}(f)(\mathfrak{H}'(X_+)^\perp) \subset \mathfrak{H}'(X_-)^\perp$, where $\mathfrak{H}'(X_\pm)^\perp$ denote the $X_\pm$-orthogonal complements of $\mathfrak{H}'(X_\pm)$ in $\mathfrak{H}(X_\pm)$.

We conclude this section by relating $\mathfrak{M}$-Hilbert spaces to partial Hilbert spaces of Gudder (1986a). Let $\rho = \{\rho_X\}_{X \in \text{Ob } \mathfrak{M}}$ be an $\mathfrak{M}$-weight and $\mathfrak{H}$ an $\mathfrak{M}$-Hilbert $\mathfrak{M}$-space. We denote by $\mathbf{H}(\mathfrak{H})$ the set of all families $\{x_X\}_{X \in \text{Ob } \mathfrak{M}}$ abiding by the following conditions:

(2.3.4) For any $X \in$ Ob $\mathfrak{M}$, $x_X \in \mathfrak{H}(X)$ and $\int_X \|x_X\|^2 \, d\rho_X < +\infty$.

(2.3.5) For any $\mathfrak{M}$-maximal Boolean locales X, Y in $\mathfrak{M}$, $\int_X \|x_X\|^2 \, d\rho_X = \int_Y \|x_Y\|^2 \, d\rho_Y$.

(2.3.6) If $\{X_\lambda \xrightarrow{f_\lambda} X\}_{\lambda \in \Lambda}$ is an $\mathfrak{M}$-coproduct diagram in $\mathfrak{M}$, then $\mathfrak{H}(f)(x_X) = x_{X_\lambda}$ for any $\lambda \in \Lambda$.

The set $\mathbf{H}(\mathfrak{H})$ can be reckoned as a partial Hilbert space with respect to the following reflexive, symmetric relation $\mathbf{S}$.

(2.3.7) $\{x_X\}_{X \in \text{Ob } \mathfrak{M}} \mathbf{S} \{y_X\}_{X \in \text{Ob } \mathfrak{M}}$ iff $\int_X \langle x_X, y_X \rangle \, d\rho_X = \int_Y \langle x_Y, y_Y \rangle \, d\rho_Y$ for all $\mathfrak{M}$-maximal Boolean locales X, Y in $\mathfrak{M}$.

3. BOUNDED OPERATORS ON EMPIRICAL HILBERT SPACES

3.1. Booleanization

Let X be a Boolean locale and $\mathscr{H}$ an X-Hilbert space. These entities shall be fixed throughout this subsection. By interpreting the notion of a bounded operator on a Hilbert space in the topos **BEns**(X), we get the notion of an *X-bounded X-operator* on $\mathscr{H}$, which is an X-linear map $\mathscr{T}$ of $\mathscr{H}$ into itself subject to the following condition:

(3.1.1) There exists $\alpha \in C_{\mathbf{R}}(X)$ with $\alpha \geq 0$ such that $\|\mathscr{T}(x)\| \leq \alpha \|x\|$ for any $x \in \mathscr{H}$.

By way of example, an X-Hilbert map of $\mathscr{H}$ into itself is an X-bounded X-operator on $\mathscr{H}$. If it is onto, then it is called an *X-unitary X-operator* on $\mathscr{H}$.

An X-bounded X-operator $\mathcal{T}$ on $\mathcal{H}$ is called X-*positive* if $\langle \mathcal{T}x, x\rangle \geq 0$ for any $x \in \mathcal{H}$. This notion is the interpretation of a positive bounded operator on a Hilbert space within the topos **BEns**(X). By using the first Boolean transfer principle, we can get the following theorem directly from Theorem 1.6.1.

Theorem 3.1.1. For any X-bounded X-positive X-operator $\mathcal{A}$ on $\mathcal{H}$, there exists a unique X-bounded X-positive X-operator $\mathcal{B}$ on $\mathcal{H}$ with $\mathcal{A} = \mathcal{B}^2$. Furthermore, $\mathcal{B}$ commutes with every X-bounded X-operator on $\mathcal{H}$ which commutes with $\mathcal{A}$.

As in standard mathematics, the above $\mathcal{B}$ is denoted by $\mathcal{A}^{1/2}$.

For any X-bounded X-operator $\mathcal{T}$ on $\mathcal{H}$ and any X-Hilbert subspace of $\mathcal{H}$ invariant under $\mathcal{T}$, $\mathcal{T}|\mathcal{K}$ denotes the restriction of $\mathcal{T}$ to $\mathcal{K}$.

Proposition 3.1.2. Let $\mathcal{A}$ be an X-bounded X-positive X-operator on $\mathcal{H}$ and $\mathcal{K}$ an X-Hilbert subspace of $\mathcal{H}$ invariant under $\mathcal{A}$. Then $\mathcal{A}^{1/2}|\mathcal{K} = (\mathcal{A}|\mathcal{K})^{1/2}$.

Proof. Booleanize Corollary 1.6.3. ■

An X-bounded X-operator $\mathcal{T}$ on $\mathcal{H}$ is called X-*self-adjoint* if $\langle \mathcal{T}x, y\rangle = \langle x, \mathcal{T}y\rangle$ for any $x, y \in \mathcal{H}$. This notion is the interpretation of a bounded self-adjoint operator on a Hilbert space in **BEns**(X). Since every bounded positive operator on a Hilbert space is always self-adjoint in standard mathematics, the first Boolean transfer principle reveals that an X-bounded X-positive X-operator is always X-self-adjoint. An X-bounded idempotent X-self-adjoint X-operator on $\mathcal{H}$ is called an X-*projection* X-*operator* on $\mathcal{H}$.

By Booleanizing the well-known bijective correspondence between the Hilbert subspaces of a Hilbert space H and the projection operators on H, we have the following result.

Theorem 3.1.3. For any X-projection X-operator $\mathcal{P}$ on $\mathcal{H}$, $\mathcal{P}(\mathcal{H})$ is an X-Hilbert subspace of $\mathcal{H}$. This gives a bijective correspondence between the X-Hilbert subspaces of $\mathcal{H}$ and the X-projection operators on $\mathcal{H}$.

We conclude this subsection simply by commenting that by interpreting the notions of the trace Tr (a function from the positive bounded operators to the extended real numbers) and a density operator (a bounded positive trace-class operator of trace one) within the topos **BEns**(X), we get the notions of Tr_X and an X-*density* X-*operator*.

3.2. Quantization

Let $\mathfrak{M}$ be a manual of Boolean locales and $\mathfrak{H}$ an $\mathfrak{M}$-Hilbert $\mathfrak{M}$-space. These entities shall be fixed throughout this subsection. An $\mathfrak{M}$-*bounded* $\mathfrak{M}$-

operator on $\mathfrak{H}$ is an assignment $\mathfrak{T}$, to each $X \in \mathrm{Ob}\,\mathfrak{M}$, of an X-bounded X-operator $\mathfrak{T}_X$ on $\mathfrak{H}(X)$ making the diagram

$$\begin{array}{ccc} & \mathfrak{h}(f) & \\ \mathfrak{h}(X_-) & \longleftarrow & \mathfrak{h}(X_+) \\ \mathfrak{T}_{X_-} \downarrow & & \downarrow \mathfrak{T}_{X_+} \\ \mathfrak{h}(X_-) & \longleftarrow & \mathfrak{h}(X_+) \\ & \mathfrak{h}(f) & \end{array}$$

commutative for any $f: X_- \to X_+ \in \mathrm{Mor}\,\mathfrak{M}$.

Given $\mathfrak{M}$-bounded $\mathfrak{M}$-operators $\mathfrak{S}$ and $\mathfrak{T}$ on $\mathfrak{H}$, the assignment $X \in \mathrm{Ob}\,\mathfrak{M} \mapsto \mathfrak{T}_X\mathfrak{S}_X$ is easily seen to be an $\mathfrak{M}$-bounded $\mathfrak{M}$-operator on $\mathfrak{H}$, which is to be denoted by $\mathfrak{T}\mathfrak{S}$. In particular, if $\mathfrak{S}$ and $\mathfrak{T}$ happen to be the same, $\mathfrak{T}\mathfrak{S}$ is denoted by $\mathfrak{S}^2$.

It is easy to see the following result.

Proposition 3.2.1. An assignment $\mathfrak{T}$, to each $X \in \mathrm{Ob}\,\mathfrak{M}$, of an X-bounded X-operator $\mathfrak{T}_X$ on $\mathfrak{H}(X)$ is an $\mathfrak{M}$-bounded $\mathfrak{M}$-operator on $\mathfrak{H}$ iff it makes the diagram

$$\begin{array}{ccc} & \mathfrak{h}(f)^\wedge & \\ \mathfrak{h}(X_-) & \longleftarrow & f^*\mathfrak{h}(X_+) \\ \mathfrak{T}_{X_-} \downarrow & & \downarrow f^*\mathfrak{T}_{X_+} \\ \mathfrak{h}(X_-) & \longleftarrow & f^*\mathfrak{h}(X_+) \\ & \mathfrak{h}(f)^\wedge & \end{array}$$

commutative for any $f: X_- \to X_+ \in \mathrm{Mor}\,\mathfrak{M}$.

An $\mathfrak{M}$-bounded $\mathfrak{M}$-operator $\mathfrak{T}$ on $\mathfrak{H}$ is said to be $\mathfrak{M}$-*positive* if $\mathfrak{T}_X$ is X-positive for any $X \in \mathrm{Ob}\,\mathfrak{M}$.

Theorem 3.2.2. For any $\mathfrak{M}$-bounded $\mathfrak{M}$-positive $\mathfrak{M}$-operator $\mathfrak{A}$ on $\mathfrak{H}$, there exists a unique $\mathfrak{M}$-bounded $\mathfrak{M}$-positive $\mathfrak{M}$-operator $\mathfrak{B}$ with $\mathfrak{B}^2 = \mathfrak{A}$.

Proof. Follows from Theorem 3.1.1 and Propositions 3.1.2 and 3.2.1. ■

An $\mathfrak{M}$-bounded $\mathfrak{M}$-operator $\mathfrak{T}$ on $\mathfrak{H}$ is said to be $\mathfrak{M}$-*self-adjoint* if $\mathfrak{T}_X$ is X-self-adjoint for any $X \in \mathrm{Ob}\,\mathfrak{M}$. An $\mathfrak{M}$-*projection* $\mathfrak{M}$-*operator* on $\mathfrak{H}$ is an $\mathfrak{M}$-bounded $\mathfrak{M}$-operator $\mathfrak{T}$ such that $\mathfrak{T}_X$ is an X-projection X-operator on $\mathfrak{H}(X)$ for each $X \in \mathrm{Ob}\,\mathfrak{M}$.

An $\mathfrak{M}$-Hilbert $\mathfrak{M}$-space $\mathfrak{H}'$ is said to be an $\mathfrak{M}$-*Hilbert* $\mathfrak{M}$-*subspace* of $\mathfrak{H}$ if it satisfies the following conditions:

(3.2.1) For any $X \in \mathrm{Ob}\,\mathfrak{M}$, $\mathfrak{H}'(X)$ is an X-Hilbert subspace of $\mathfrak{H}(X)$.
(3.2.2) For any $f: X_- \to X_+ \in \mathrm{Mor}\,\mathfrak{M}$, $\mathfrak{H}(f)(\mathfrak{H}'(X_+)) \subseteq \mathfrak{H}'(X_-)$, $\mathfrak{H}(f)(\mathfrak{H}'(X_+)^\perp) \subseteq \mathfrak{H}'(X_-)^\perp$, and $\mathfrak{H}'(f)$ is the restriction of $\mathfrak{H}(f)$.

It is easy to see the following result.

Theorem 3.2.3. For any $\mathfrak{M}$-Hilbert $\mathfrak{M}$-subspace $\mathfrak{H}'$ of $\mathfrak{H}$, the assignment, to each $X \in \mathrm{Ob}\,\mathfrak{M}$, of the X-projection X-operator corresponding to the X-Hilbert subspace $\mathfrak{H}'(X)$ of $\mathfrak{H}(X)$ in Theorem 3.1.3 is an $\mathfrak{M}$-projection $\mathfrak{M}$-operator, which gives a bijective correspondence between the $\mathfrak{M}$-Hilbert $\mathfrak{M}$-subspaces of $\mathfrak{H}$ and the $\mathfrak{M}$-projection $\mathfrak{M}$-operators on $\mathfrak{H}$.

4. UNBOUNDED OPERATORS ON EMPIRICAL HILBERT SPACES

4.1. Booleanization

Let X be a Boolean locale and $\mathscr{H}$ an X-Hilbert space. These entities shall be fixed throughout this subsection. An X-linear map $\mathscr{A}$ from an X-linear subspace $\mathscr{D}(\mathscr{T})$ of $\mathscr{H}$ to $\mathscr{H}$ is called an X-*self-adjoint* X-*operator* on $\mathscr{H}$ if the X-orthogonal complement of the graph $\Gamma(\mathscr{A})$ of $\mathscr{A}$ in $\mathscr{H} \oplus \mathscr{H}$ is $\{(-\mathscr{A}x, x) \mid x \in \mathscr{D}(\mathscr{A})\}$. It is easy to see that an X-bounded X-operator $\mathscr{A}$ is X-self-adjoint in this sense iff it is so in the sense of Section 3.

By Booleanizing the notion of a spectral measure, we get the notion of an X-*spectral* X-*measure*, which is a function $\mathscr{E}$ on $\mathscr{B}_0(\mathbf{R})$ subject to the following conditions:

(4.1.1) $\mathscr{E}(M)$ is an X-projection X-operator on $\mathscr{H}$ for each $M \in \mathscr{B}_0(\mathbf{R})$.
(4.1.2) $\mathscr{E}(\mathbf{R}) = \mathscr{I}$, where $\mathscr{I}$ is the identity X-operator on $\mathscr{H}$.
(4.1.3) $\mathscr{E}(M \cup N) = \mathscr{E}(M) + \mathscr{E}(N)$ for any disjoint sets M and N in $\mathscr{B}_0(\mathbf{R})$.
(4.1.4) If $\{M_n\}_{n\in\mathbf{N}}$ is an increasing sequence of sets in $\mathscr{B}_0(\mathbf{R})$ such that $\cup^{X}_{n\in\mathbf{N}} M_n$ is also in $\mathscr{B}_0(\mathbf{R})$, then $\mathscr{E}(\cup^{X}_{n\in\mathbf{N}} M_n) = \mathrm{LUB}_{n\in\mathbf{N}}\, \mathscr{E}(M_n)$, where $\mathrm{LUB}_{n\in\mathbf{N}}\, \mathscr{E}(M_n)$ stands for the least upper bound of $\mathscr{E}(M_n)$'s within the lattice of all X-projection X-operators on $\mathscr{H}$.

The measure $d\langle x, E_\lambda x\rangle$ and the integral $\int_{-\infty}^{+\infty}$ in Section 1.6 are Booleanized to yield the X-measure $d\langle x, \mathscr{E}_\lambda x\rangle$ and the X-integral X-$\int_{-\infty}^{+\infty}$. By Booleanizing Theorem 1.6.5, we have the following result.

Theorem 4.1.1. Each X-spectral X-measure $\mathscr{E}$ determines a unique X-self-adjoint X-operator on $\mathscr{H}$, denoted by X-$\int_{-\infty}^{+\infty}\lambda\, d\mathscr{E}_\lambda$, satisfying the following conditions:

(4.1.5) $\mathscr{D}(\text{X-}\int_{-\infty}^{+\infty}\lambda\ d\mathscr{E}_\lambda) = \{x \in \mathscr{H} \mid \text{X-}\int_{-\infty}^{+\infty}\lambda^2\ d\langle x, \mathscr{E}_\lambda x\rangle \in C_{\mathbf{R}}(\text{X})\}$.

(4.1.6) $\langle x, (\text{X-}\int_{-\infty}^{+\infty}\lambda\ d\mathscr{E}_\lambda)x\rangle = \text{X-}\int_{-\infty}^{+\infty}\lambda\langle x, \mathscr{E}_\lambda x\rangle$ for any $x \in \mathscr{D}(\text{X-}\int_{-\infty}^{+\infty}\lambda\ d\mathscr{E}_\lambda)$.

The correspondence $\mathscr{E} \mapsto \text{X-}\int_{-\infty}^{+\infty}\lambda\ d\mathscr{E}_\lambda$ gives a bijective correspondence between the X-spectral X-measures and the X-self-adjoint X-operators on $\mathscr{H}$.

A mapping $\mathfrak{A}$ assigning, to each $t \in \mathbf{R}$, an X-unitary X-operator $\mathfrak{A}(t)$ on $\mathscr{H}$ is called a *one-parameter* X-*unitary group* on $\mathscr{H}$ if it satisfies the following conditions:

(4.1.7) $\mathfrak{A}(s + t) = \mathfrak{A}(s)\mathfrak{A}(t)$ for all $s, t \in \mathbf{R}$.

(4.1.8) For any $x \in \mathscr{H}$, any natural number n, and any positive number ϵ, there exist a partition $\{e_\lambda\}_{\lambda\in\Lambda}$ of unity of $\mathbf{B}$ and a family $\{\delta_\lambda\}_{\lambda\in\Lambda}$ of positive numbers such that for any $\lambda \in \Lambda$ and any $s, t \in [-n, n]$, whenever $|s - t| < \delta_\lambda$, $\|e_\lambda\mathfrak{A}(s)x - e_\lambda\mathfrak{A}(t)x\| < \epsilon$

By Theorem 1.6.6 and the well-known fact of classical mathematics that every continuous function from a compact metric space to a metric space is uniformly continuous, the first Boolean transfer principle gives at once the following result.

Theorem 4.1.2. Let $\mathscr{A}$ be an X-self-adjoint X-operator on $\mathscr{H}$ and $\mathfrak{A}(t) = \exp_{\text{X}}(it\mathscr{A})$ for each $t \in \mathbf{R}$, where the function $\exp_{\text{X}}$ stands simply for the interpretation of the well-known function exp of classical mathematics within the topos **BEns**(X). Then $\mathfrak{A}$ is a one-parameter X-unitary group on $\mathscr{H}$.

By using the first Boolean transfer principle and recalling the well-known fact of classical mathematics that any uniformly continuous function from a dense subset of a metric space into another metric space can be extended uniquely to a continuous function defined on the former metric space, we get from Theorem 1.6.7 the following result.

Theorem 4.1.3. Let $\mathfrak{A}$ be a one-parameter X-unitary group on $\mathscr{H}$. Then there exists an X-self-adjoint X-operator $\mathscr{A}$ on $\mathscr{H}$ with $\mathfrak{A}(t) = \exp_{\text{X}}(it\mathscr{A})$ for any $t \in \mathbf{R}$. The desired $\mathscr{A}$ is defined for $x \in \mathscr{H}$ iff $\text{X-lim}_{t\to 0}\ t^{-1}(\mathfrak{A}(t)x - x)$ exists, in which we set

$$\mathscr{A}x = i^{-1}\ \text{X-lim}_{t\to 0}\ t^{-1}(\mathfrak{A}(t)x - x)$$

4.2. Quantization

Let $\mathfrak{M}$ be a manual of Boolean locales and $\mathfrak{H}$ an $\mathfrak{M}$-Hilbert $\mathfrak{M}$-space. These entities shall be fixed throughout this subsection. An $\mathfrak{M}$-*self-adjoint* $\mathfrak{M}$-*operator* on $\mathfrak{H}$ is an assignment $\mathfrak{A}$, to each $\text{X} \in \mathfrak{M}$, of an X-self-adjoint

X-operator $\mathfrak{A}_X$ such that for any f: $X_- \to X_+ \in$ Mor $\mathfrak{M}$ and any $x \in \mathscr{D}(\mathfrak{A}_{X_+})$, we have $\mathfrak{H}(f)x \in \mathscr{D}(\mathfrak{A}_{X_-})$ and $\mathfrak{A}_{X_-}\mathfrak{H}(f)x = \mathfrak{H}(f)\mathfrak{A}_{X_+}x$.

An *$\mathfrak{M}$-spectral $\mathfrak{M}$-measure* is a function $\mathfrak{E}$ assigning, to each X $\in$ Ob $\mathfrak{M}$, an X-spectral X-measure $\mathfrak{E}^X$ such that:

(4.2.1) For each $M \in \mathscr{B}_0(\mathbf{R})$, the assignment X $\in$ Ob $\mathfrak{M} \mapsto \mathfrak{E}^X(M)$ is an $\mathfrak{M}$-projection $\mathfrak{M}$-operator.

Now we have an empirical spectral theorem as follows:

Theorem 4.2.1. Given an $\mathfrak{M}$-spectral $\mathfrak{M}$-measure $\mathfrak{E}$, the assignment X $\in$ Ob $\mathfrak{M} \mapsto$ X-$\int_{-\infty}^{+\infty}\lambda\, d\mathfrak{E}_\lambda^X$ is an $\mathfrak{M}$-self-adjoint $\mathfrak{M}$-operator on $\mathfrak{H}$, which is denoted symbolically by $\mathfrak{M}$-$\int_{-\infty}^{+\infty}\lambda\, d\mathfrak{E}_\lambda$. The correspondence $\mathfrak{E} \mapsto \mathfrak{M}$-$\int_{-\infty}^{+\infty}\lambda\, d\mathfrak{E}_\lambda$ gives a bijective correspondence between the $\mathfrak{M}$-spectral $\mathfrak{M}$-measures and the $\mathfrak{M}$-self-adjoint $\mathfrak{M}$-operators on $\mathfrak{H}$.

Proof. Follows from Theorem 4.1.1. ■

A mapping $\mathfrak{U}$ assigning, to each X $\in$ Ob $\mathfrak{M}$, a one-parameter X-unitary group $\mathfrak{U}_X$ on $\mathfrak{H}(X)$ is called a *one-parameter $\mathfrak{M}$-unitary group* on $\mathfrak{H}$ if for any $t \in \mathbf{R}$, the assignment X $\in$ Ob $\mathfrak{M} \mapsto \mathfrak{U}_X(t)$ is a unitary $\mathfrak{M}$-operator on $\mathfrak{H}$.

It is easy to see the following.

Lemma 4.2.2. Let $\{\mathfrak{H}_n\}_{n\in\mathbf{N}}$ be an increasing sequence of Hilbert $\mathfrak{M}$-subspaces of $\mathfrak{H}$ and $\mathfrak{T}$ an assignment, to each X $\in$ Ob $\mathfrak{M}$, of an X-bounded X-operator on $\mathfrak{H}(X)$. Assume that:

(4.2.2) For each $n \in \mathbf{N}$ and each X $\in$ Ob $\mathfrak{M}$, $\mathfrak{H}_n(X)$ is invariant under $\mathfrak{T}_X$.

(4.2.3) For each $n \in \mathbf{N}$, the assignment X $\in$ Ob $\mathfrak{M} \mapsto \mathfrak{T}_X | \mathfrak{H}_n(X)$ is an $\mathfrak{M}$-bounded $\mathfrak{M}$-operator on $\mathfrak{H}_n$.

(4.2.4) For each X $\in$ Ob $\mathfrak{M}$, $\cup_{n\in\mathbf{N}}^X \mathfrak{H}_n(X)$ is X-dense in $\mathfrak{H}(X)$.

Then the assignment X $\in$ Ob $\mathfrak{M} \mapsto \mathfrak{T}_X$ is an $\mathfrak{M}$-bounded $\mathfrak{M}$-operator on $\mathfrak{H}$.

Theorem 4.2.3. Let $\mathfrak{A}$ be an $\mathfrak{M}$-self-adjoint $\mathfrak{M}$-operator on $\mathfrak{H}$ and $\mathfrak{U}_X(t) = \exp_X(it\mathfrak{A}_X)$ for any X $\in$ Ob $\mathfrak{M}$ and any $t \in \mathbf{R}$. Then $\mathfrak{U}$ is a one-parameter $\mathfrak{M}$-unitary group on $\mathfrak{H}$.

Proof. Use Lemma 4.2.2, in which $\mathfrak{H}_n$ should be taken to be the $\mathfrak{M}$-Hilbert $\mathfrak{M}$-subspace corresponding to the $\mathfrak{M}$-projection $\mathfrak{M}$-operator X $\in$ Ob $\mathfrak{M} \mapsto \mathfrak{E}^X([-n, n])$ for each $n \in \mathbf{N}$. ■

Theorem 4.2.4. Let $\mathfrak{U}$ be a one-parameter $\mathfrak{M}$-unitary group on $\mathfrak{H}$. Then there exists an $\mathfrak{M}$-self-adjoint $\mathfrak{M}$-operator $\mathfrak{A}$ on $\mathfrak{H}$ with $\mathfrak{U}_X(t) = \exp_X(it\mathfrak{A}_X)$ for any X $\in$ Ob $\mathfrak{M}$ and any $t \in \mathbf{R}$.

Proof. Using the second Boolean transfer principle, it is easy to see that for any $\mathfrak{f}: \mathrm{X}_- \to \mathrm{X}_+ \in \mathrm{Mor}\ \mathfrak{M}$ and any $x \in \mathfrak{H}(\mathrm{X}_+)$, if

$$\mathrm{X}\text{-}\lim_{t^+ \to 0} t^{-1}(\mathfrak{A}_{\mathrm{X}_+}(t)x - x)$$

exists, then

$$\mathrm{X}\text{-}\lim_{t^- \to 0} t^{-1}(\mathfrak{A}_{\mathrm{X}_-}(t)\mathfrak{H}(\mathfrak{f})x - \mathfrak{H}(\mathfrak{f})x)$$

exists and

$$\mathrm{X}\text{-}\lim_{t^- \to 0} t^{-1}(\mathfrak{A}_{\mathrm{X}_-}(t)\mathfrak{H}(\mathfrak{f})x - \mathfrak{H}(\mathfrak{f})x)$$
$$= \mathfrak{H}(\mathfrak{f})(\mathrm{X}\text{-}\lim_{t^+ \to 0} t^{-1}(\mathfrak{A}_{\mathrm{X}_+}(t)x - x))$$

from which and Theorem 4.1.3 the desired result follows readily. ∎

5. QUANTUM MECHANICS

5.1. Standard Quantum Mechanics

We now recall the skeletal framework of standard quantum mechanics. The state space of a strictly quantum system is usually assumed to be represented by a Hilbert space H. The states are represented by density operators on H. Physical quantities are represented by (possibly unbounded) self-adjoint operators on H. The expectation value of a physical quantity Q in a state S is given by $\mathrm{Tr}(QS)$. The dynamics of the system is governed by a one-parameter unitary group $t \in \mathbf{R} \mapsto U(t)$ in the sense that the state $S(t_1)$ at time t_1 and the state $S(t_2)$ at time t_2 are related by:

(5.1.1) $S(t_2) = U(t_2 - t_1)S(t_1)U(t_2 - t_1)^{-1}$

By Stone's theorem there exists a self-adjoint operator A with $U(t) = \exp(-itA)$ for all $t \in \mathbf{R}$. Therefore, if the state of the system at an instant is pure so that the state of the system at time t is represented by a vector $x(t)$ in H, the infinitesimal form of the dynamics of the system is depicted by the following Schrödinger equation:

(5.1.2) $i(d/dt)x(t) = Ax(t)$.

5.2. Boolean Quantum Mechanics

Machida and Namiki (1980) have devised a framework for the interaction between classical and quantum systems in which a continuous superselection rule plays an important role and the so-called reduction of the wave packet is nicely handled. Their many-Hilbert-spaces formalism for concurrent classical and quantum systems was refined mathematically by Araki (1980) from an operator-algebraic viewpoint. He considered the algebra of observables of such a combined system to be generally of a nontrivial center, the operators of which behave like classical observables. Finally Ozawa (1986) replaced the direct integral of Hilbert spaces by a more general "Boolean Hilbert space." As we demonstrated amply in Nishimura (1993a), the Boolean-valued analysis of Takeuti (1978) and others can be regarded as a direct descendant of direct integral theory.

Now we will explain briefly Ozawa's (1986) Boolean quantum mechanics based on Boolean Hilbert space theory. Let X be a Boolean locale. The state space is represented by an X-Hilbert space $\mathscr{H}$. The states are represented by X-density X-operators on $\mathscr{H}$. Physical quantities are represented by X-self-adjoint X-operators on $\mathscr{H}$. The expectation value of a physical quantity $\mathscr{Q}$ in a state $\mathscr{S}$ within the topos **BEns**(X) is given by $\mathrm{Tr}_{\mathrm{X}}\mathscr{Q}\mathscr{S})$. The dynamics of the system is governed by a one-parameter X-unitary group $t \in \mathbf{R} \mapsto \mathfrak{A}(t)$ in the sense that the state $\mathscr{S}(t_1)$ at time t_1 and the state $\mathscr{S}(t_2)$ at time t_2 are connected by

$$(5.2.1) \quad \mathscr{S}(t_2) = \mathfrak{A}(t_2 - t_1)\mathscr{S}(t_1)\mathfrak{A}(t_2 - t_1)^{-1}.$$

By Theorem 4.1.2 there exists an X-self-adjoint X-operator $\mathscr{A}$ with $\mathfrak{A}(t) = \exp(-it\mathscr{A})$ for all $t \in \mathbf{R}$. Therefore, if the state of the system at an instant is pure so that the state of the system at time t is represented by a vector $x(t)$ in $\mathscr{H}$, the infinitesimal form of the dynamics of the system is depicted by the following Schrödinger equation:

$$(5.2.2) \quad i(d^{\mathrm{X}}/d^{\mathrm{X}}t)x(t) = \mathscr{A}x(t), \text{ where}$$

$$\frac{d^{\mathrm{X}}}{d^{\mathrm{X}}t}x(t) = \mathrm{X}\text{-}\lim_{t'\to 0}\frac{x(t + t') - x(t)}{t'}$$

5.3. Empirical Quantum Mechanics

If the measuring apparatus is no longer classical but quantum so that it is represented not by a single Boolean locale X but by a manual $\mathfrak{M}$ of Boolean locales, we are naturally led to empirical quantum mechanics. The state space is now represented by an $\mathfrak{M}$-Hilbert $\mathfrak{M}$-space $\mathfrak{H}$. The states are represented by X-density X-operators on $\mathfrak{H}(\mathrm{X})$ for all $\mathrm{X} \in \mathrm{Ob}\,\mathfrak{M}$. Physical quantities are represented by $\mathfrak{M}$-self-adjoint $\mathfrak{M}$-operators on $\mathfrak{H}$. The expectation value of a physical quantity $\mathfrak{Q}$ in a state represented by an X-density X-operator $\mathscr{S}$ within the topos **BEns**(X) is given by $\mathrm{Tr}_{\mathrm{X}}(\mathfrak{Q}_{\mathrm{X}}\mathscr{S})$. The dynamics of the system is governed by a one-parameter $\mathfrak{M}$-unitary group $\mathfrak{U}$ in the sense that the state at time t_1 represented by an X-density X-operator $\mathcal{S}(t_1)$ and the state at time t_2 represented by an X-density X-operator $\mathcal{S}(t_2)$ are related by

$$(5.3.1)\quad \mathcal{S}(t_2) = \mathfrak{U}_X(t_2 - t_1)\mathcal{S}(t_1)\mathfrak{U}_X(t_2 - t_1)^{-1}.$$

By Theorem 4.2.3 there exists an $\mathfrak{M}$-self-adjoint $\mathfrak{M}$-operator $\mathfrak{A}$ with $\mathfrak{U}_X(t) = \exp_X(-it\mathfrak{A}_X)$ for all $X \in$ Ob $\mathfrak{M}$ and all $t \in \mathbf{R}$. Therefore, if the state of the system at an instant is pure so that the state of the system at time t is represented by a vector $x(t)$ in $\mathfrak{H}(X)$, the infinitesimal form of the dynamics of the system is depicted by the following Schrödinger equation:

$$(5.3.2)\quad i(d^X/d^Xt)x(t) = \mathfrak{A}_X x(t)$$

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